F P3 June 13 (R) M.A Kprime 2

1. The hyperbola H has foci at (5, 0) and (–5, 0) and directrices with equations

$x = \frac{9}{5}$ and $x = -\frac{9}{5}$.

Find a cartesian equation for H.

(7)

1. $\pm ae = \pm 5$

$ae = 5 \Rightarrow a = \frac{5}{e} \quad e = \frac{5}{a}$

Directrices $\Rightarrow \frac{9}{5} = \frac{a}{e}$, $-\frac{9}{5} = -\frac{a}{e}$

$\therefore \frac{9}{5} = \frac{a}{5/a} = \frac{a^2}{5}$

$\therefore a^2 = 9 \Rightarrow a = 3$

$\therefore 3e = 5 \Rightarrow e = \frac{5}{3}$

$\therefore \frac{x^2}{9} - \frac{y^2}{16} = 1$ $\quad a=3, b=4$

$b^2 = |a^2(1-e^2)| \Rightarrow b^2 = 16 \Rightarrow b = 4$

~~$\therefore b^2 = 9(1 - \frac{25}{9}) < 0$~~

2. Two skew lines l_1 and l_2 have equations

$$l_1: \mathbf{r} = (\mathbf{i} - \mathbf{j} + \mathbf{k}) + \lambda(4\mathbf{i} + 3\mathbf{j} + 2\mathbf{k})$$
$$l_2: \mathbf{r} = (3\mathbf{i} + 7\mathbf{j} + 2\mathbf{k}) + \mu(-4\mathbf{i} + 6\mathbf{j} + \mathbf{k})$$

respectively, where λ and μ are real parameters.

(a) Find a vector in the direction of the common perpendicular to l_1 and l_2 **(2)**

(b) Find the shortest distance between these two lines. **(5)**

2(a). $\begin{pmatrix}4\\3\\2\end{pmatrix} \times \begin{pmatrix}-4\\6\\1\end{pmatrix} = \begin{matrix} \times & 4 & 3 & 2 & 4 & 3 & 2 \\ & -4 & 6 & 1 & -4 & 6 & 1 \end{matrix}$

$= \begin{pmatrix}-9\\-12\\36\end{pmatrix}$

$\therefore \hat{} = \begin{pmatrix}-9\\-12\\36\end{pmatrix}$ //

(b) ~~d=~~ ~~d=~~ $d = \left| \dfrac{\left(\begin{pmatrix}1\\-1\\1\end{pmatrix} - \begin{pmatrix}3\\7\\2\end{pmatrix}\right) \cdot \begin{pmatrix}-9\\-12\\36\end{pmatrix}}{\left|\begin{pmatrix}-9\\-12\\36\end{pmatrix}\right|} \right|$

$\begin{pmatrix}-2\\-8\\-1\end{pmatrix}$

$\therefore d = \dfrac{78}{39} \Rightarrow d = 2$ //

3. The point P lies on the ellipse E with equation

$$\frac{x^2}{36} + \frac{y^2}{9} = 1$$

N is the foot of the perpendicular from point P to the line $x = 8$

M is the midpoint of PN.

(a) Sketch the graph of the ellipse E, showing also the line $x = 8$ and a possible position for the line PN.

(1)

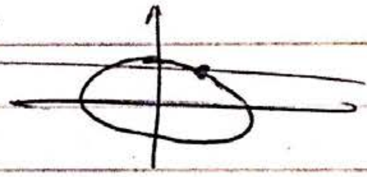
(b) Find an equation of the locus of M as P moves around the ellipse.

(4)

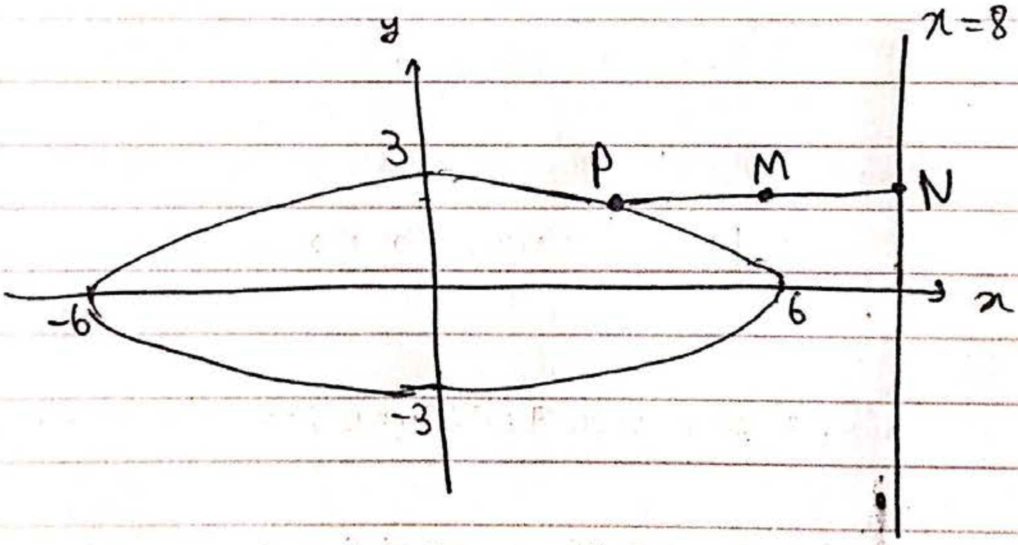
(c) Show that this locus is a circle and state its centre and radius.

(3)

~~3(a).~~



3(a).



$\cancel{36 + e^2}$ -

$\cancel{= 36(1 - e^2)}$

(b)

(b) ~~y=k~~

$P(a\cos\theta, b\sin\theta)$

$W(8, b\sin\theta)$

$\therefore M: \left(\dfrac{8+a\cos\theta}{2}, b\sin\theta\right)$

$\therefore x = \dfrac{8+a\cos\theta}{2} \Rightarrow \dfrac{2x-8}{a} = \cos\theta$

$y = b\sin\theta \Rightarrow \sin\theta = \dfrac{y}{b}$

$\therefore \sin^2\theta + \cos^2\theta = 1 = \dfrac{y^2}{b^2} + \dfrac{(2x-8)^2}{a^2}$

$\therefore \dfrac{(2x-8)^2}{a^2} + \dfrac{y^2}{b^2} = 1$

~~$x^2 + y^2 = r^2$~~

~~$a = 6$~~ $a^2 = 36$

$b^2 = 9$

$$\therefore \frac{(2x-8)^2}{36} + \frac{y^2}{9} = 1$$

(c) ~~$\frac{1}{9}(x-4)^2 + \frac{1}{9}y^2 = 1$~~

$$\frac{(2x-8)^2}{36} + \frac{y^2}{9} = \frac{[2(x-4)]^2}{36} + \frac{y^2}{9}$$

$$= \frac{4}{36}(x-4)^2 + \frac{y^2}{9}$$

$$= \frac{1}{9}(x-4)^2 + \frac{y^2}{9} = 1$$

$$\therefore (x-4)^2 + y^2 = 9$$

$\therefore$ Centre is $(4, 0)$

radius $= 3$

4. The plane Π_1 has vector equation

$$\mathbf{r} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + s\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix},$$

where s and t are real parameters.

The plane Π_1 is transformed to the plane Π_2 by the transformation represented by the matrix $\mathbf{T}$, where

$$\mathbf{T} = \begin{pmatrix} 2 & 0 & 3 \\ 0 & 2 & -1 \\ 0 & 1 & 2 \end{pmatrix}$$

Find an equation of the plane Π_2 in the form $\mathbf{r}.\mathbf{n} = p$

(9)

Π_1

$$r = \begin{pmatrix} 1+s+t \\ -1+s+2t \\ 2-2t \end{pmatrix}$$

$$\therefore \begin{pmatrix} 2 & 0 & 3 \\ 0 & 2 & -1 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} 1+s+t \\ -1+s+2t \\ 2-2t \end{pmatrix}$$

$$= \begin{pmatrix} 2+2s+2t+6-6t \\ -2+2s+4t-2+2t \\ -1+s+2t+4-4t \end{pmatrix}$$

$$= \begin{pmatrix} 8+2s-4t \\ -4+2s+6t \\ 3+s-2t \end{pmatrix}$$

$$= \begin{pmatrix} 8 \\ -4 \\ 3 \end{pmatrix} + s\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} + t\begin{pmatrix} -4 \\ 6 \\ -2 \end{pmatrix}$$

$$\therefore \underset{\sim}{n} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \times \begin{pmatrix} -4 \\ 6 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} -10 \\ 0 \\ 20 \end{pmatrix}$$

$$\Rightarrow \underset{\sim}{r} \cdot \underset{\sim}{n} = \begin{pmatrix} 8 \\ -4 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} -10 \\ 0 \\ 20 \end{pmatrix} = -20$$

$$\underset{\sim}{r} \cdot 10 \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} = -20$$

$$\Rightarrow \underset{\sim}{r} \cdot \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} = -2$$

5.

$$I_n = \int_1^5 x^n (2x-1)^{-\frac{1}{2}}\,dx, \quad n \geqslant 0$$

(a) Prove that, for $n \geqslant 1$,

$$(2n+1)I_n = nI_{n-1} + 3 \times 5^n - 1$$

(5)

(b) Using the reduction formula given in part (a), find the exact value of I_2

(5)

5(a). $I_n = \int_1^5 x^n (2x-1)^{-1/2}\,dx$

Let $u = x^n \qquad u' = nx^{n-1}$

$v' = (2x-1)^{-1/2} \qquad v = \frac{1}{2}(2x-1)^{1/2} \cdot 2$

$= (2x-1)^{1/2}$

$\therefore I_n = \left[x^n (2x-1)^{1/2}\right]_1^5 - n\int_1^5 x^{n-1}(2x-1)^{1/2}\,dx$

$\therefore I_n = 3(5^n) - 1 - n\int_1^5 x^{n-1}(2x-1)^{1/2}$

$(2x-1)^{1/2} = (2x-1)^{-1/2}(2x-1)$

$\therefore I_n = 3(5^n) - 1 - n\int_1^5 x^{n-1}(2x-1)(2x-1)^{-1/2}\,dx$

$\therefore I_n = 3(5^n) - 1 - n\int 2x^n(2x-1)^{-1/2} - x^{n-1}(2x-1)^{-1/2}\,dx$

$\therefore\ I_n = 3(5^n) - 1 - n\left(2I_n - I_{n-1}\right)$

$\therefore\ I_n = 3(5^n) - 1 - 2nI_n + nI_{n-1}$

$\therefore\ \cancel{\#}\ (2n+1)I_n = nI_{n-1} + 3(5^n) - 1$

$\Rightarrow\ (2n+1)I_n = nI_{n-1} + 3\times5^n - 1$

as required.

6) $\cancel{I_2}$ using $n=2 \Rightarrow$

$5I_2 = 2I_1 + 74$

$\therefore\ \cancel{5I_2 = 2\int_1^5 x}$

using $n=1 \Rightarrow$

$3I_1 = I_0 + 14$

$\therefore\ 3I_1 = \int_1^5 (2x-1)^{-1/2}\,dx + 14$

$\therefore\ 3I_1 = \left[(2x-1)^{1/2}\right]_1^5 + 14$

$\therefore\ 3I_1 = 16 \Rightarrow I_1 = \frac{16}{3}$

$\therefore\ 5I_2 = 2\times\frac{16}{3}\ \cancel{+17}\ + 74 \Rightarrow I_2 = \frac{254}{15}$

6. It is given that $\begin{pmatrix}1\\2\\0\end{pmatrix}$ is an eigenvector of the matrix **A**, where

$$\mathbf{A} = \begin{pmatrix}4 & 2 & 3\\2 & b & 0\\a & 1 & 8\end{pmatrix}$$

and a and b are constants.

(a) Find the eigenvalue of **A** corresponding to the eigenvector $\begin{pmatrix}1\\2\\0\end{pmatrix}$. **(3)**

(b) Find the values of a and b. **(3)**

(c) Find the other eigenvalues of **A**. **(5)**

6 (a). $A\underline{x} = \lambda \underline{x}$

$$\therefore \cancel{\begin{pmatrix}4 & 2 & 3\\2 & b & 0\\a & 1 & 8\end{pmatrix} = \begin{pmatrix}\lambda\\2\lambda\\0\end{pmatrix}}$$

$\Rightarrow$

$$\therefore \begin{pmatrix}4 & 2 & 3\\2 & b & 0\\a & 1 & 8\end{pmatrix}\begin{pmatrix}1\\2\\0\end{pmatrix} = \begin{pmatrix}\lambda\\2\lambda\\0\end{pmatrix}$$

$$\therefore \begin{pmatrix}8\\2+2b\\a+2\end{pmatrix} = \begin{pmatrix}\lambda\\2\lambda\\0\end{pmatrix} \Rightarrow \lambda = 8$$ is the corresponding E. Value

Question 6 continued

(b) $Ax = \lambda x$

$$\Rightarrow \begin{pmatrix} 8 \\ 2+2b \\ a+2 \end{pmatrix} = \begin{pmatrix} 8 \\ 16 \\ 0 \end{pmatrix}$$

$\therefore \quad a+2=0 \Rightarrow a=-2$

$2+2b=16 \Rightarrow b=7$

(c) $A-\lambda I = \begin{pmatrix} 4-\lambda & 2 & 3 \\ 2 & 7-\lambda & 0 \\ -2 & 1 & 8-\lambda \end{pmatrix}$

$\det(A-\lambda I)=0 \Rightarrow \cancel{(4-\lambda)(7-\lambda)(8-\lambda)-2(2(8-\lambda))+3(}$

$(4-\lambda)(7-\lambda)(8-\lambda)-2(2(8-\lambda))+3(2+2(7-\lambda))=0$

$\det(A-\lambda I)=(4-\lambda)(7-\lambda)(8-\lambda)-4(8-\lambda)+6(7-\lambda)+6=0$

$\cancel{\lambda=8 \Rightarrow \det(A-\lambda I)=0}$ $\quad 6(7-\lambda+1)$

$\therefore \cancel{(4-\lambda)} \; \cancel{(8-\lambda)[(4-\lambda)(7-\lambda)-4]+6(7-\lambda)+6=0}$

$\therefore (4-\lambda)(7-\lambda)(8-\lambda)-4(8-\lambda)+6(8-\lambda)=0$

$\therefore (8-\lambda)[(4-\lambda)(7-\lambda)+2]$ $\qquad \lambda^2-11\lambda+30=0$

$\therefore (8-\lambda)(\lambda^2-11\lambda+30)=0 \Rightarrow$ $\cancel{\therefore \lambda = \frac{11 \pm \sqrt{\ }}{2}}$

$\lambda^2-11\lambda+30 \Rightarrow (\lambda-5)(\lambda-6)=0$ ~~via~~ $\rightarrow$

$\therefore \lambda=5, \lambda=6$ ~~Quadratic Formula~~

7.

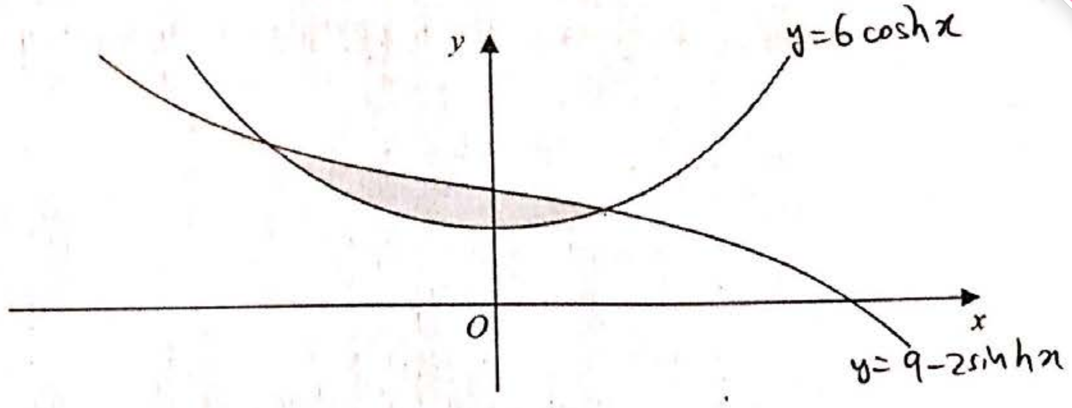


Figure 1

The curves shown in Figure 1 have equations

$$y = 6\cosh x \quad \text{and} \quad y = 9 - 2\sinh x$$

(a) Using the definitions of sinh x and cosh x in terms of e^x, find exact values for the x-coordinates of the two points where the curves intersect.

(6)

The finite region between the two curves is shown shaded in Figure 1.

(b) Using calculus, find the area of the shaded region, giving your answer in the form $a \ln b + c$, where a, b and c are integers.

(6)

7(a). $y = 6\cosh x$ & $y = 9 - 2\sinh x$ @ intersection

$\therefore \quad 6\cosh x = 9 - 2\sinh x$

$\Rightarrow \cancel{6}\ \cancel{3(e}\ 3e^x + 3e^{-x} = 9 - (e^x - e^{-x})$

$\therefore \quad 3e^x + 3e^{-x} = 9 - e^x + e^{-x}$

$\therefore \quad 4e^x + 2e^{-x} - 9 = 0$

$\times e^x \quad \therefore \quad 4e^{2x} - 9e^x + 2 = 0$

$(e^x - 2)(4e^x - 1) = 0$

$\Rightarrow e^x = 2 \Rightarrow x = \ln 2$

$e^x = 1/4 \Rightarrow x = \ln \frac{1}{4}$ //

(b) Area $= \int_{\ln \frac{1}{4}}^{\ln 2} 9 - 2\sinh x \, \partial x - \int_{\ln 1/4}^{\ln 2} 6\cosh x \, \partial x$

$\therefore$ Area $= \int_{\ln 1/4}^{\ln 2} 9 - 2\sinh x - 6\cosh x \, \partial x$

$= \left[9x - 2\cosh x - 6\sinh x \right]_{\ln 1/4}^{\ln 2}$

$= 9\ln 2 - 2\cosh[\ln(2)] - 6\sinh[\ln 2]$

$\cancel{\frac{1}{4} = (2^2)^{-1}}$

$- 9\ln 1/4 + 2\cosh(\ln \frac{1}{4}) + 6\sinh(\ln \frac{1}{4})$

$\frac{1}{4} = (2^2)^{-1}$

~~Use $\ln(\frac{1}{4}) = -2\ln(2)$~~

$\therefore$ ~~$= 9\ln 2 - 2\cosh(\ln 2) - 6\sinh(\ln 2)$~~

$= 9\ln 2 - 2 \cdot \frac{2 + 2^{-1}}{2} - \frac{6}{2}(2 - 2^{-1}) - 9\ln\frac{1}{4} + \frac{1}{4} + 4$

$+ 3(1/4 - 4)$

$= 9\ln 2 - 2 - \frac{1}{2} - 6 + \frac{3}{2} - 9\ln\frac{1}{4} + \frac{1}{4} + 4 + \frac{3}{4} - 12$

$= -14 + 9\ln 8$

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8.

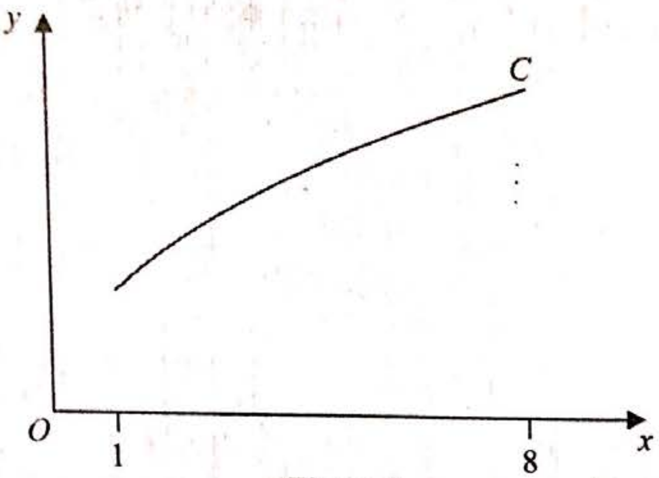

Figure 2

The curve C, shown in Figure 2, has equation

$$y = 2x^{\frac{1}{2}}, \qquad 1 \leqslant x \leqslant 8$$

(a) Show that the length s of curve C is given by the equation

$$s = \int_1^8 \sqrt{\left(1 + \frac{1}{x}\right)} \, dx$$

(2)

(b) Using the substitution $x = \sinh^2 u$, or otherwise, find an exact value for s.

Give your answer in the form $a\sqrt{2} + \ln(b + c\sqrt{2})$ where a, b and c are integers.

(9)

8(a) $y = 2x^{1/2} \therefore \frac{dy}{dx} = x^{-1/2} = \frac{1}{\sqrt{x}} \therefore \left(\frac{dy}{dx}\right)^2 = \frac{1}{x} = \left(\frac{1}{\sqrt{x}}\right)^2$

sub in $\left(\frac{dy}{dx}\right)^2$

$s = \int_1^8 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx = \int_1^8 \sqrt{1 + \frac{1}{x}} \, dx$

as required.

Question 8 continued

(b) $x = \sinh^2 u$

First sort out limits:

$8 = \sinh^2 u \Rightarrow \quad \sinh u = \sqrt{8}$

$\therefore \quad u = \operatorname{arsinh}(\sqrt{8}) = \cancel{\ln(\sqrt{8} + \quad)}$

$\ln(\sqrt{8} + 3)$

$1 = \sinh^2 u \Rightarrow \sinh u = 1$

$\therefore \quad u = \operatorname{arsinh}(1) = \ln(1 + \sqrt{2})$

New limits are $\int_{\ln(1+\sqrt{2})}^{\ln(3+\sqrt{8})}$

$$\sqrt{1 + \frac{1}{x}} = \sqrt{1 + \frac{1}{\sinh^2 u}} = \sqrt{\frac{\sinh^2 u}{\sinh^2 u} + \frac{1}{\sinh^2 u}}$$

$c^2 - s^2 = 1$

$$= \sqrt{\frac{\sinh^2 u + 1}{\sinh^2 u}} = \sqrt{\frac{\cosh^2 u}{\sinh^2 u}}$$

$$= \coth(u)$$

$$\therefore \quad \sqrt{1 + \frac{1}{x}} = \coth u$$

$x = \sinh^2 u$

$\therefore \frac{dx}{du} = 2\sinh u \cosh u$

$\therefore dx = 2\sinh u \cosh u \ du$

Now:

$\therefore \sqrt{1+\frac{1}{x}} = \coth u \quad \& \quad dx = 2\sinh u \cosh u \, du$

$\therefore S = \int_1^8 \sqrt{1+\frac{1}{x}} \, dx = 2\int_{\ln(1+\sqrt{2})}^{\ln(3+\sqrt{8})} \frac{\cosh u}{\sinh u} \cdot \sinh u \cdot \cosh u \, du$

$= 2\int_{\ln(1+\sqrt{2})}^{\ln(3+\sqrt{8})} \cosh^2 u \, du$

$\cancel{c^2}$

$c^2 + s^2 = \cosh 2x$

$\cancel{c^2+1}$ $\quad c^2 + c^2 - 1$

$2c^2 - 1$

$c^2 - s^2 = 1$

$c^2 - 1$

$= \cancel{2}\int_{\ln(1+\sqrt{2})}^{\ln(3+\sqrt{8})} \cosh(2x) + 1 \, dx$

$= \left[\frac{1}{2}\sinh 2x + x\right]_{\ln(1+\sqrt{2})}^{\ln(3+\sqrt{8})}$

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Question 8 continued

$$= \frac{1}{2}\sinh\left(2\ln(3+\sqrt{8})\right) + \ln(3+\sqrt{8}) - \frac{1}{2}\sinh\left(\ln(1+\sqrt{2})^2\right) - \ln(1+\sqrt{2})$$

$$= \ln(1+\sqrt{2}) + \frac{1}{2}\sinh\left(2\ln(3+\sqrt{8})\right) - \frac{1}{2}\sinh\left(\ln(1+\sqrt{2})^2\right)$$

$$= \ln(1+\sqrt{2}) + \frac{1}{2}\left(\frac{e^{\ln[(3+\sqrt{8})^2]} - e^{-\ln[(3+\sqrt{8})^2]}}{2}\right)$$

$$- \frac{1}{2}\left(\frac{e^{\ln(1+\sqrt{2})^2} - e^{-\ln(1+\sqrt{2})^2}}{2}\right)$$

$$= \ln(1+\sqrt{2}) + \frac{1}{2}\left(\frac{(3+\sqrt{8})^2 - (3+\sqrt{8})^{-2}}{2}\right)$$

$$- \frac{1}{2}\left(\frac{(1+\sqrt{2})^2 - (1+\sqrt{2})^{-2}}{2}\right)$$

7-95

Question 8 continued

$$= \ln(1+\sqrt{2}) + \frac{1}{4}\left(17+12\sqrt{2} - 17 + 12\sqrt{2}\right)$$

$$- \frac{1}{4}\left(3+2\sqrt{2} - 3 + 2\sqrt{2}\right)$$

$\frac{1}{(1+\sqrt{2})^2}$

$$= \ln(1+\sqrt{2}) + 6\sqrt{2} - \sqrt{2}$$

$$= 5\sqrt{2} + \ln(1+\sqrt{2})$$